

EIGENVALUES OF ELLIPTIC BOUNDARY VALUE PROBLEMS WITH AN INDEFINITE WEIGHT FUNCTION

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ABSTRACT. We consider general selfadjoint elliptic eigenvalue problems

$$(P) \quad \mathcal{A}u = \lambda r(x)u,$$

in an open set $\Omega \subset \mathbf{R}^k$. Here, the operator \mathcal{A} is positive and of order $2m$ and the “weight” r is a function which changes sign in Ω and is allowed to be discontinuous. A scalar λ is said to be an eigenvalue of (P) if $\mathcal{A}u = \lambda ru$ —in the variational sense—for some nonzero u satisfying the appropriate growth and boundary conditions. We determine the asymptotic behavior of the eigenvalues of (P), under suitable assumptions. In the case when Ω is bounded, we assumed Dirichlet or Neumann boundary conditions. When Ω is unbounded, we work with operators of “Schrödinger type”; if we set $r_{\pm} = \max(\pm r, 0)$, two cases appear naturally: First, if Ω is of “weighted finite measure” (i.e., $\int_{\Omega}(r_{+})^{k/2m} < +\infty$ or $\int_{\Omega}(r_{-})^{k/2m} < +\infty$), we obtain an extension of the well-known Weyl asymptotic formula; secondly, if Ω is of “weighted infinite measure” (i.e., $\int_{\Omega}(r_{+})^{k/2m} = +\infty$ or $\int_{\Omega}(r_{-})^{k/2m} = +\infty$), our results extend the de Wet-Mandl formula (which is classical for Schrödinger operators with weight $r \equiv 1$). When Ω is bounded, we also give lower bounds for the eigenvalues of the Dirichlet problem for the Laplacian.

1. Introduction. The purpose of this paper is to provide estimates on the eigenvalues of “right nondefinite” elliptic boundary value problems $\mathcal{A}u = \lambda ru$ in $\Omega \subset \mathbf{R}^k$ (with Dirichlet or Neumann boundary conditions) when the weight function r is “indefinite” (i.e., changes sign in Ω).

The main results concern the asymptotics of the eigenvalues for selfadjoint elliptic operators of order $2m$. For an open set Ω which is not necessarily bounded, we extend the classical Weyl formula as well as its analogue, also called the de Wet-Mandl formula for operators of Schrödinger type. We also obtain lower bounds for the eigenvalues of the Dirichlet problem for the Laplacian. These results extend the earlier ones obtained independently by Lapidus [La 1, La 2] and Fleckinger-El Fetnassi [FIF 1, FIF 2].

Many nonlinear problems lead, after linearization, to elliptic eigenvalue problems with an indefinite weight function. (See, e.g., the survey paper by de Figueiredo

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[dF] and the work of Hess and Kato [HK, H].) A vast literature in engineering, physics and applied mathematics deals with such problems arising, for instance, in the study of transport theory, reaction-diffusion equations, fluid dynamics, etc. (See, e.g., [DGHa, GN, LuR, LuW and KKLeZ].)

Most of the literature on this subject—studied since the end of the last century (Richardson [Ri], Bôcher [Bo], Hilbert [Hi], etc.)—is concerned with the one dimensional case, which is still the object of active research. (See, for instance, [KZ, Be].) To our knowledge, the first result in the multidimensional case is due to Holmgren (1904) [Ho]; he considers the Dirichlet problem $\Delta u + \lambda r(x, y)u = 0$, on a bounded open set $\Omega \subset \mathbf{R}^2$, when r is continuous and changes sign; he proves in this case the existence of an infinite number of positive (and negative) eigenvalues which can be characterized by the “min-max principle”. The asymptotic distribution of these eigenvalues has been established by Pleijel (1942) [Pl] for the Dirichlet and Neumann problems.

The spectral theory of similar abstract problems has been studied more recently by Weinberger [Wn], and its application to the Dirichlet problem for an elliptic operator of order 2 by Manes and Micheletti [MM].

Pleijel’s results on the asymptotic distribution of eigenvalues have been extended by many Soviet mathematicians during the 1970s. (See, for instance, the survey paper by Birman and Solomjak [BS 3], their book [BS 4] and the relevant references therein.) In most of these papers, the weight function need not be continuous. Birman and Solomjak [BS 1, BS 2], for example, obtain asymptotic estimates for bounded open sets and (possibly degenerate) selfadjoint elliptic operators of order $2m$. Rozenbljum [Ro 1] extends these results to some problems defined on unbounded open sets such that $\int_{\Omega} |r|^{k/2m} < +\infty$.

Recently, unaware of Pleijel’s work, Lapidus [La 1] determined the asymptotic behavior of the eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions on a bounded open set of \mathbf{R}^k in the case when the weight function is not necessarily continuous; he also gives lower bounds having the correct “coupling constant behavior” for the eigenvalues of the Dirichlet problem. In some cases, he obtains estimates on the remainder term [La 2].

At about the same time, Fleckinger and El Fetnassi [FIF 1, FIF 2] proved an analogous asymptotic estimate for the Dirichlet problem corresponding to a general selfadjoint elliptic operator of order $2m$ with a continuous weight function; asymptotics were also obtained for operators of “Schrödinger-type” on unbounded domains for a sufficiently smooth weight function.

The present paper extends these results: for general elliptic problems, we do not assume that the open set Ω is bounded or that the weight function r is continuous; the latter is of interest in view of possible applications to nonlinear problems.

We now indicate how this work is organized. After having stated our main results in the next section, we present in §3 the prototypal example of the Laplacian on an open set of finite volume and give lower bounds for the eigenvalues of the corresponding Dirichlet problem. We then extend this study in §4 to general elliptic boundary value problems of order $2m$ on an open set Ω (which is bounded for the Neumann problem). Finally, in §5, we determine the asymptotic distribution of the eigenvalues of certain Schrödinger operators on $\Omega \subset \mathbf{R}^k$ with Dirichlet boundary

conditions; in this case, Ω is unbounded and we deal with the four different cases: $\int_{\Omega}(r_{\pm})^{k/2m} = +\infty$ or $\int_{\Omega}(r_{\pm})^{k/2m} < +\infty$.

Since the first version of this paper was submitted, the work of Birman-Solomjak and Rozenbljum on boundary value problems with indefinite weights was brought to our attention. In particular, there is some overlap between Theorem 4.1 herein and Theorem 2 in [BS 1 or BS 2], as well as Theorem 5.2 herein and Theorem 2 in [Ro 1]. However, our results go much further; Theorem 5.1 and Proposition 5.4, stated for unbounded open sets, appear to be entirely new as well as the possible combinations resulting from Theorems 5.1 and 5.2.

2. Notation and main results. We introduce some notation which will be used throughout the article:

Let m and k be integers ≥ 1 .

Let Ω be an open set in \mathbf{R}^k with boundary $\partial\Omega$.

The interior (resp., the closure) of $Q \subset \mathbf{R}^k$ is denoted by Q° (resp., \overline{Q}); if, in addition, Q is Lebesgue measurable, $|Q|$ stands for its Lebesgue k -dimensional measure or “volume” in \mathbf{R}^k .

If $R \subset Q$, the notation $Q \setminus R$ indicates the complement of R in Q .

For $j \in \mathbf{N}$, $C^j(\Omega)$ is the space of j times continuously differentiable functions in Ω .

For $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{N}^k$, D^α is the derivative of order $|\alpha| = \alpha_1 + \dots + \alpha_k$: $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}$; if, in addition, $\xi = (\xi_1, \dots, \xi_k) \in \mathbf{R}^k$, then $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_k^{\alpha_k}$.

$H^m(\Omega)$ denotes the Sobolev space of functions $u \in L^2(\Omega)$ such that the distributional derivatives $D^\alpha u$ are in $L^2(\Omega)$ for $|\alpha| \leq m$ (see, e.g., [Ad]). Recall that $H^m(\Omega)$ is a Hilbert space for the norm

$$\|u\|_{H^m(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha| \leq m} |D^\alpha u(x)|^2 dx \right)^{1/2}.$$

$H_0^m(\Omega)$ is the subspace of $H^m(\Omega)$ obtained by completing $C_0^\infty(\Omega)$ with respect to the norm of $H^m(\Omega)$; here, $C_0^\infty(\Omega)$ denotes the space of continuously differentiable functions with compact support contained in Ω .

Finally, if f is a real-valued function defined on Ω , we set $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$.

We can now state our hypotheses and present our main results:

(2.1) Let \mathcal{A} be a formally selfadjoint uniformly elliptic operator of order $2m$ defined on Ω :

$$\mathcal{A} = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^\alpha (a_{\alpha\beta}(x) D^\beta),$$

where $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$ for $|\alpha| \leq m$ and $|\beta| \leq m$; we assume that $a_0 \geq 0$ and—when Ω is bounded (resp., unbounded)— $a_0 \in L^{k/2m}(\Omega)$ if $k \geq 2m$ and $a_0 \in L^1(\Omega)$ if $k < 2m$ [resp., $a_0 \in L^\infty(\Omega)$]; moreover, $a_{\alpha\beta} \in L^\infty(\Omega)$ if $0 < |\alpha| + |\beta| \leq 2m$ and $a_{\alpha\beta} \in C^0(\overline{\Omega})$ if $|\alpha| + |\beta| = 2m$.

(2.2) Let r be a measurable real-valued function on Ω ; we suppose that $\Omega_+ = \{x \in \Omega: r(x) > 0\}$ and $\Omega_- = \{x \in \Omega: r(x) < 0\}$ are of positive Lebesgue measure. In §§3–4 (resp., §5), we also assume that $r \in L^p(\Omega)$ [resp., $r \in L_{\text{loc}}^p(\Omega)$] with $p > k/2m$ if $k \geq 2m$ and $p = 1$ if $k < 2m$. Further, in §§3–4, Ω is such that the imbedding of

$H_0^m(\Omega)$ into $L^2(\Omega)$ is compact; this is always the case if Ω is bounded, $k < 2m$ or $\partial\Omega$ is smooth enough (see [Ad, Theorem 6.13, p. 151]).

We study the spectrum (i.e., the set of eigenvalues λ) of the homogeneous Dirichlet or Neumann boundary value problem (in the variational sense)

$$(P) \quad \mathcal{A}u = \lambda ru, \quad \text{in } \Omega.$$

When Ω has finite Lebesgue measure, this spectrum is discrete and consists of a double sequence of real eigenvalues of finite multiplicity (one nonnegative and one negative)

$$\cdots \leq \lambda_{n+1}^- \leq \lambda_n^- \leq \cdots \leq \lambda_2^- \leq \lambda_1^- < (0) \leq \lambda_1^+ \leq \lambda_2^+ \leq \cdots \leq \lambda_n^+ \leq \lambda_{n+1}^+ \leq \cdots,$$

with $|\lambda_n^\pm|$ tending to $+\infty$ as n tends to ∞ (each eigenvalue is repeated according to multiplicity). The same result holds, under appropriate assumptions, for an operator of ‘‘Schrödinger type’’ on Ω with Dirichlet boundary conditions.

Let $N^+(\lambda)$ [resp., $N^-(\lambda)$] be the number of nonnegative [resp., negative] eigenvalues λ_n^+ less than $\lambda > 0$ [resp., λ_n^- greater than $\lambda < 0$].

Assume that Ω has weighted finite volume. (Of course, this assumption is satisfied if Ω is bounded.) If $|\Omega_+ \setminus \Omega_+^\circ| = 0$ (this is the case, for example, if r is continuous), we show for the Dirichlet problem that (see Theorem 4.1)

$$(2.3) \quad N^+(\lambda) \sim \int_{\Omega_+} (\lambda r(x))^{k/2m} \mu'_{\mathcal{A}}(x) dx, \quad \text{as } \lambda \rightarrow +\infty,$$

where $\mu'_{\mathcal{A}}(x)$ is the usual Browder-Gårding density on Ω

$$\mu'_{\mathcal{A}}(x) = (2\pi)^{-k} \int_{\{\xi \in \mathbf{R}^k: \mathcal{A}'(x, \xi) < 1\}} d\xi, \quad x \in \Omega,$$

and $\mathcal{A}'(x, \xi)$ is the symbol of the principal part of \mathcal{A} :

$$\mathcal{A}'(x, \xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha+\beta}, \quad \text{for } (x, \xi) \in \Omega \times \mathbf{R}^k.$$

Similarly, if $|\Omega_- \setminus \Omega_-^\circ| = 0$, then

$$(2.4) \quad N^-(\lambda) \sim \int_{\Omega_-} (\lambda r(x))^{k/2m} \mu'_{\mathcal{A}}(x) dx, \quad \text{as } \lambda \rightarrow -\infty.$$

In particular, when $\mathcal{A} = -\Delta$, we have

$$\mu'_{\mathcal{A}}(x) = (2\pi)^{-k} B_k,$$

where $B_k = \pi^{k/2}/\Gamma(1 + k/2)$ denotes the volume of the unit ball in \mathbf{R}^k ; since $N^+(\lambda_n^+) = n$, we then obtain an extension of Hermann Weyl’s asymptotic formula (see Theorems 3.1 and 3.2):

$$(2.5) \quad |\lambda_n^\pm| \sim C_k n^{2/k} \|r_\pm\|_{L^{k/2}(\Omega)}^{-1}, \quad \text{as } n \rightarrow \infty,$$

where $C_k = (2\pi)^2 (B_k)^{-2/k}$ denotes Weyl’s constant.

REMARKS. 1. Actually, (2.5) contains two equations: one involving the minus signs alone and a second one for the plus signs; we shall use this convenient sign convention throughout the paper.

2. Note that the results for negative eigenvalues can be deduced directly from those for positive eigenvalues by changing r into $-r$.

3. For simplicity, we write Ω_{\pm}° instead of $(\Omega_{\pm})^{\circ}$.

If, in addition, Ω is a bounded open set satisfying the segment property (for example, if $\partial\Omega$ is locally Lipschitz) and if $|\partial\Omega| = 0$, the formulae (2.3) and (2.4) still hold for the Neumann problem (see Theorem 4.2).

When $r(x) \equiv 1$, $N^-(\lambda) = 0$ and (2.3) is the usual formula (see, e.g., [CH, Chapter VI, §4, pp. 429–445; Br; Ga; Kc; FlMe; Me; RS, Theorem XIII.78, p. 271; etc.]).

When Ω is unbounded, we assume that \mathcal{A} is an operator of “Schrödinger type”, i.e., $\mathcal{A} = \mathcal{L} + q$, where \mathcal{L} satisfies (2.1).

(2.6) Here, the “potential” q is a nonnegative function on Ω such that $q^{-1}(x)r(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.

Under other suitable hypotheses, we show that the spectrum of (P) is discrete. Moreover, we prove that (see Theorem 5.1 and Proposition 5.4)

$$(2.7) \quad N^{\pm}(\lambda) \sim \int_{\Omega_{\pm} \cap \Omega_{\lambda}} (\lambda r(x) - q(x))^{k/2m} \mu'_{\mathcal{L}}(x) dx, \quad \text{as } \lambda \rightarrow \pm\infty,$$

if $\int_{\Omega} (r_{\pm})^{k/2m} = +\infty$, respectively; here, $\Omega_{\lambda} = \{x \in \Omega: \lambda r(x) > q(x)\}$ and $\mu'_{\mathcal{L}} = \mu'_{\mathcal{A}}$, by definition.

This estimate is well-known when $r(x) \equiv 1$. (See, e.g., [T, Chapter 17, §§8–14, pp. 174–185; Ro 2; Me; Fl; RS, Theorem XIII.81, p. 275; etc.].)

If $\int_{\Omega} (r_{\pm})^{k/2m} < +\infty$, then estimate (2.3) or (2.4) holds respectively as $\lambda \rightarrow \pm\infty$. (See Theorem 5.2 and Remark 1 following it as well as Proposition 5.4.)

We also establish lower bounds for the eigenvalues of the Dirichlet problem on a bounded domain Ω ; for instance, for the Laplacian and for $k \geq 3$, we have (see Theorem 3.3)

$$(2.8) \quad |\lambda_n^{\pm}| \geq \theta_k n^{2/k} \|r_{\pm}\|_{L^{k/2}(\Omega)}^{-1},$$

where θ_k is a constant depending only on k which is explicitly known. In view of the extended Weyl formula (2.5), these lower bounds exhibit the correct “coupling constant behavior”.

We refer the reader to Courant-Hilbert [CH, Chapter VI], Reed-Simon [RS, Chapter XIII, §§1, 15] or Weinberger [Wn, Chapter 3] for the Courant-Weyl method of proof of Weyl’s formula. For the variational theory of elliptic boundary value problems, we point out Lions-Magenes [LM] and Agmon [Ag]. Finally, the basic properties of Sobolev spaces used in this paper can all be found in Adams [Ad].

3. An example: the Laplacian on an open set with finite volume. We now illustrate our methods for finding the asymptotic behavior of the eigenvalues, in the case of the Laplacian with Dirichlet or Neumann boundary conditions on an open set with finite volume or on a bounded open set, respectively (parts A and B). In part C, we obtain lower bounds for the eigenvalues of the corresponding Dirichlet problem on a bounded domain.

A. *The Dirichlet problem.* We consider the eigenvalue problem

$$(E) \quad -\Delta u = \lambda r u, \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega.$$

(3.1) Ω is an open subset of \mathbf{R}^k with finite Lebesgue measure, satisfying (2.2) and such that $|\Omega_+ \setminus \Omega_+^\circ| = 0$ and $|\Omega_- \setminus \Omega_-^\circ| = 0$.

The eigenvalues of the variational problem associated with (E) are the complex numbers λ such that there exists a nonzero $u \in H_0^1(\Omega)$ satisfying $-\Delta u = \lambda r u$ (in the distributional sense).

It is well known (see, e.g., [Ho; Pl; Wn, Chapter 3; MM; dF, Chapter I, §§1–2; etc.]) that the eigenvalues λ_n^\pm (when they exist) are characterized by the “max-min principle”

$$(3.2) \quad \frac{1}{\lambda_n^\pm} = \max_{F_n \in \mathcal{F}_n} \min_{u \in F_n} \left\{ \int_{\Omega} r|u|^2 : \|\text{grad } u\|_{L^2(\Omega)}^2 = 1 \right\},$$

where \mathcal{F}_n denotes the set of n -dimensional subspaces of $H_0^1(\Omega)$; an analogous formula holds for λ_n^- .

The hypothesis $|\Omega_\pm| > 0$ implies that λ_n^\pm exists for all $n \geq 1$ and $\lambda_n^\pm \rightarrow \pm\infty$ as $n \rightarrow \infty$. The precise asymptotic behavior is given by

THEOREM 3.1. *Under assumptions (2.2) and (3.1), estimates (2.3)–(2.5) hold.*

If r is positive in Ω , we have $N^-(\lambda) = 0$ (actually, this happens if and only if $|\Omega_-| = 0$). Formulae (2.3) and (2.5)—which are clearly equivalent—are classical when r is positive and continuous (see, e.g., [CH, Theorem 14, p. 435; RS, Theorem XIII.78, p. 271; Me, Theorem 5.12, p. 188; etc.]).

We shall need two lemmas in order to derive Theorem 3.1. The following result is essentially known and for $k \geq 3$, for instance, can be obtained by combining [RS, Theorem XIII.80, p. 274] and [LiY, Step (iv), pp. 317–318]. (For the case $k = 1$, see [GoK, Theorem 8.1, p. 309].)

LEMMA 3.1. *When r is positive in $L^p(\Omega)$ with $p > k/2$ if $k \geq 3$ and $p = 1$ if $k < 3$, (2.3) and (2.5) hold.*

In the following, we shall denote by $\lambda_n^+(r, \Omega)$ the n th nonnegative eigenvalue of (E) to indicate the dependence on r and Ω .

LEMMA 3.2 (MONOTONICITY PRINCIPLES). *$\lambda_n^+(r, \Omega)$ does not increase when r or Ω increases.*

REMARKS. 1. Here and thereafter, we adopt the following convention: whenever we make use of an eigenvalue λ_n^+ , we implicitly assume that it exists (equivalently, that the right-hand side of (3.2)—or of its counterpart—is finite).

2. The monotonicity with respect to Ω is only valid for Dirichlet problems.

Lemma 3.2 is a simple consequence of (3.2) and is well-known when r is positive (see, e.g., [CH, Theorem 3, p. 409 and Theorem 7, p. 411; RS, Proposition 4, p. 270 and Wn, Theorem 7.1, p. 58, Example 7.2, p. 60 and Theorem 8.1, p. 62].

PROOF OF THEOREM 3.1. We follow the method used in [La 1, La 2]: the idea consists in finding lower and upper bounds for λ_n^+ having the same asymptotics. With this aim and the help of the monotonicity principles, we reduce the problem to the study of positive weights.

It clearly suffices to establish (2.5) for λ_n^+ since $\lambda_n^-(r, \Omega) = -\lambda_n^+(-r, \Omega)$ and $r_- = (-r)_+$.

Fix $\delta > 0$. By Lemma 3.2, we have

$$(3.3) \quad \lambda_n^+(r_+ + \delta, \Omega) \leq \lambda_n^+(r, \Omega) \leq \lambda_n^+(r, \Omega_+^\circ);$$

this holds since $r \leq r_+ + \delta$ and Ω_+° is an open subset of Ω of positive measure, by (2.2) and (3.1).

By Lemma 3.1 applied to the positive weights $r_+ + \delta$ in Ω and r in Ω_+° , we see that

$$(3.4) \quad \lim_{n \rightarrow \infty} n^{-2/k} \lambda_n^+(r_+ + \delta, \Omega) = C_k \|r_+ + \delta\|_{L^{k/2}(\Omega)}^{-1}$$

and

$$(3.5) \quad \lim_{n \rightarrow \infty} n^{-2/k} \lambda_n^+(r, \Omega_+^\circ) = C_k \|r\|_{L^{k/2}(\Omega_+^\circ)}^{-1}.$$

We infer from equations (3.3) to (3.5) that

$$(3.6) \quad \begin{aligned} C_k \|r_+ + \delta\|_{L^{k/2}(\Omega)}^{-1} &\leq \varliminf_{n \rightarrow \infty} n^{-2/k} \lambda_n^+(r, \Omega) \\ &\leq \varlimsup_{n \rightarrow \infty} n^{-2/k} \lambda_n^+(r, \Omega) \leq C_k \|r\|_{L^{k/2}(\Omega_+^\circ)}^{-1}. \end{aligned}$$

Now, it follows from the Lebesgue dominated convergence theorem that $\|r_+ + \delta\|_{L^{k/2}(\Omega)}$ tends to $\|r\|_{L^{k/2}(\Omega_+^\circ)}$ as δ tends to 0; moreover, by (3.1),

$$\int_{\Omega_+^\circ} r^{k/2} = \int_{\Omega_+} r^{k/2} = \int_{\Omega} (r_+)^{k/2}.$$

[Note that by omitting the zero set of r , one does not alter the value of the latter integral.]

Consequently, we obtain (2.5) by letting δ tend to zero in (3.6). \square

REMARKS. 1. When r is continuous, Ω_+ is open and Theorem 3.1 holds for an arbitrary open subset Ω of \mathbf{R}^k with finite Lebesgue measure (in particular, for any bounded open set).

2. When Ω_+ is a “Jordan contented set” (roughly speaking, if Ω_+ is bounded and well approximated from within and without by a finite union of cubes—see, e.g., [LoS, Chapter 8, §§6–7; RS, p. 271; Pe, Chapter 2, §§11–12]), we have $|\Omega_+ \setminus \Omega_+^\circ| = 0$ since then, by [LoS, Proposition 6.1, p. 332], $|\partial(\Omega_+)| = 0$; this was the assumption of [La 1, Theorem 1, p. 266].

B. *The Neumann problem.* We suppose that

(3.7) Ω is a bounded open set satisfying the segment property (see [Ag, Definition 2.1, p. 11]) such that $|\partial\Omega| = 0$, $|\Omega_+ \setminus \Omega_+^\circ| = 0$ and $|\Omega_- \setminus \Omega_-^\circ| = 0$.

We consider the eigenvalue problem

$$(E') \quad -\Delta u = \lambda r u \quad \text{in } \Omega; \quad \partial u / \partial n = 0 \quad \text{on } \partial\Omega;$$

where $\partial / \partial n$ denotes the normal derivative.

The scalar λ is said to be an eigenvalue of the variational problem associated with (E') if $-\Delta u = \lambda r u$ for some nonzero $u \in H^1(\Omega)$.

The corresponding nonnegative eigenvalues $\Lambda_n^+(r, \Omega)$ are, in this case, given by (3.2), where $H_0^1(\Omega)$ is replaced by $H^1(\Omega)$.

Since $H_0^1(\Omega) \subset H^1(\Omega)$, it follows from the “max-min principle” as in [CH, Theorem 5, p. 410; RS, Proposition 4, p. 270 or Wn, Theorem 7.1, p. 58 and Example 7.4, pp. 61–62] that

$$(3.8) \quad \Lambda_n^+(r, \Omega) \leq \lambda_n^+(r, \Omega);$$

and, as above, the eigenvalues are monotone with respect to the weight; therefore, for $\delta > 0$, we have

$$(3.9) \quad \Lambda_n^+(r_+ + \delta, \Omega) \leq \Lambda_n^+(r, \Omega) \leq \lambda_n^+(r, \Omega).$$

As for the Dirichlet problem, using an extension of (2.3) or (2.5) when Ω satisfies (3.7) and the weight is positive [**FIme**; **Me**, Theorem 5.12, p. 188], we obtain

THEOREM 3.2. *Under assumptions (2.1), (2.2) and (3.7), estimates (2.3)–(2.5) hold.*

Next, we comment on possible extensions or interpretations of the above results:

REMARKS. 1. The hypothesis concerning the segment property in Theorem 3.2 can be weakened as in [**Me**, Condition (C'), p. 156].

2. It is noteworthy that no assumption has been made about $\Omega_0 = \{x \in \Omega: r(x) = 0\}$ in order to obtain Theorems 3.1 and 3.2.

3. It is easy to derive from Theorems 3.1 and 3.2 a corresponding result for mixed Dirichlet-Neumann boundary conditions; naturally, the asymptotic distribution of the eigenvalues remains unchanged in this case.

4. The conclusions of Theorems 3.1 or 3.2 can be interpreted as follows: the positive (resp., negative) eigenvalues of (E) or (E') have the same asymptotic behavior as the eigenvalues—in the usual sense—of the elliptic operator $-\Delta/r(x)$ [resp., $+\Delta/r(x)$], defined in Ω_+° (resp., Ω_-°) for the corresponding boundary conditions. Here, we use implicitly the fact that $\|r_\pm\|_{L^{k/2}(\Omega)} = \|r\|_{L^{k/2}(\Omega_\pm^\circ)}$.

[The preceding remarks apply, with obvious changes, to the general elliptic boundary value problems studied in §4.B.]

5. Physically, when r is positive, the eigenvalues of (E) represent the natural frequencies of a vibrating membrane fixed along its boundary $\partial\Omega$ and of mass density equal to r —as well as of tension unity. (See, e.g., [**CH**, Chapters V and VI or **Kc**].) Therefore, in the case when r changes sign, we may interpret Theorem 3.1 by saying that the large values of λ_n^+ (resp., λ_n^-) are determined by the “positive mass” (resp., “negative mass”) distribution of the “membrane”.

EXAMPLE 3.1. We now consider a simple example of discontinuous weight function. Naturally, our assumptions would allow us to treat much more singular weights.

Let Ω be an open subset of \mathbf{R}^k with finite measure $|\Omega|$. Let Ω_+ and Ω_- be two disjoint measurable subsets of Ω of positive measure such that $|\Omega_+ \setminus \Omega_+^\circ| = 0$ and $|\Omega_- \setminus \Omega_-^\circ| = 0$. We define

$$r(x) = \begin{cases} -1 & \text{if } x \in \Omega_-, \\ 0 & \text{if } x \in \Omega \setminus (\Omega_- \cup \Omega_+), \\ +1 & \text{if } x \in \Omega_+. \end{cases}$$

We then deduce from Theorem 3.1 that, for the Dirichlet problem, we have

COROLLARY 3.1. $\lambda_n^\pm \sim C_k(n/|\Omega_\pm|)^{2/k}$, as $n \rightarrow \infty$.

The same estimate holds for the Neumann problem under the additional assumptions that $|\partial\Omega| = 0$ and Ω is bounded and satisfies the segment property.

REMARK. The geometric features of the above example suggest the following natural extension of the well-known isospectral problem (see, e.g., [**Kc** and **Y**, pp.

23–24]): let $\Omega^1, \Omega_{\pm}^1, r_1$ and $\Omega^2, \Omega_{\pm}^2, r_2$, be defined as in Example 3.1. Assume that the problem (E) [or (E')] has the same set of eigenvalues, counting multiplicities, in Ω_1 and Ω_2 . Under which (generic) conditions does there exist an isometry which sends Ω^1, Ω_{+}^1 and Ω_{-}^1 onto, respectively, Ω^2, Ω_{+}^2 and Ω_{-}^2 ? In other words, to what extent does the spectrum determine the sign of the weight?

C. *Lower bounds of the eigenvalues.* We obtain here lower bounds (resp., upper) bounds for the eigenvalues λ_n^+ (resp., λ_n^-) of the Dirichlet problem (E) studied in part A. These bounds are valid for all n and compatible with the asymptotic behavior obtained in Theorem 3.1.

THEOREM 3.3. *Assume that Ω is a bounded domain of \mathbf{R}^k with C^2 boundary $\partial\Omega$ and suppose that $k \geq 3$. Then, if $|\Omega_{\pm}| > 0$, respectively, we have for the Dirichlet problem (E) in Ω*

$$(3.10) \quad |\lambda_n^{\pm}| \geq \theta_k n^{2/k} \|r_{\pm}\|_{L^{k/2}(\Omega)}^{-1}, \quad \forall n \geq 1,$$

where $\theta_k = (k(k-2)/4e)(\omega_{k-1})^{2/k}$, $e = \exp(1)$ and $\omega_{k-1} = kB_k$ denotes the area of the unit sphere in \mathbf{R}^k .

PROOF. It clearly suffices to find a lower bound for λ_n^+ when $|\Omega_+| > 0$. Fix $n \geq 1$ and $\delta > 0$. Since $r \leq r_+ + \delta$, we derive from Lemma 3.2 that

$$(3.11) \quad \lambda_n^+(r, \Omega) \geq \lambda_n^+(r_+ + \delta, \Omega).$$

By [LiY, Theorem 2, p. 314] applied to the positive weight $r_+ + \delta$, we have

$$(3.12) \quad \lambda_n^+(r_+ + \delta, \Omega) \geq \theta_k n^{2/k} \|r_+ + \delta\|_{L^{k/2}(\Omega)}^{-1}.$$

Combining (3.11) and (3.12), we now obtain (3.10) by letting $\delta \rightarrow 0$ and applying the dominated convergence theorem. \square

Since $N^+(\lambda_n^+) = n$, we deduce

COROLLARY 3.2.

$$\left(\frac{k(k-2)}{4e} \right)^{k/2} \omega_{k-1} N^{\pm}(\lambda) \leq \lambda^{k/2} \left(\int_{\Omega} (r_{\pm})^{k/2} \right), \quad \forall \lambda > 0.$$

We have just used a remarkable result of Li and Yau [LiY, Theorem 2, p. 314] which improved significantly the existing upper bounds for the number of “bound states” of the Schrödinger operator in \mathbf{R}^3 . (See [LiY, p. 311 and Lb, p. 243] for references to previous works on this subject.) Note, however, that the method of Li and Yau does not seem *a priori* to apply to indefinite weights.

REMARKS. 1. The assumption that $|\Omega_{\pm} \setminus \Omega_{\pm}^0| = 0$ is not needed for Theorem 3.3 to hold.

2. We note that upper bounds for $|\lambda_n^{\pm}|$ can be deduced similarly from the known upper bounds for the eigenvalues of Dirichlet problems with positive weights. Indeed, by (3.5), $|\lambda_n^{\pm}(r, \Omega)| \leq |\lambda_n^{\pm}(r, \Omega_{\pm}^0)|$.

3. As is the case with the result of Li and Yau, Theorem 3.3 extends to a compact Riemannian manifold with C^2 boundary; θ_k must then be replaced by a constant depending on the Sobolev constant of the manifold.

4. Theorem 3.3 also extends to general selfadjoint elliptic operators of order 2; in this case, the new constant occurring in (3.10) depends on θ_k and the constant

of uniform ellipticity of the operator. This is easily seen by using a monotonicity argument (based on Lemma 4.2) and Theorem 3.3.

5. It may be worth pointing out that, for $k \geq 3$, Theorem 3.3 provides us with a new proof of the fact that λ_n^\pm exists for all $n \geq 1$ and $\lambda_n^\pm \rightarrow \pm\infty$ when $|\Omega_\pm| > 0$. (Compare with [MM, Proof of Proposition 3, p. 290 or dF, Proof of Proposition 1.11, p. 43].)

4. Elliptic boundary value problems with bounded coefficients. We extend the results of §§3.A and 3.B to selfadjoint elliptic operators of order $2m$.

A. *Abstract theory.* In order to obtain results on more general boundary value problems, we briefly explain the abstract theory (see, e.g., [Wn, Chapter 3 and dF, Chapter I]). It will later be applied to operators with bounded (§4.B) as well as unbounded (§5) coefficients (that is, in the latter case, Schrödinger operators).

(4.1) Let H be a real or complex Hilbert space.

(4.2) Let (V, H, a) be a variational triple: V is a dense subspace of H with continuous imbedding and a is a hermitian, bounded and coercive form defined on V (see [LM]).

(4.3) b is a hermitian form on V such that, for some positive constant δ ,

$$|b(u, u)| \leq \delta a(u, u), \quad \forall u \in V.$$

We consider the following variational eigenvalue problem

$$(Q) \quad a(u, v) = \lambda b(u, v), \quad \forall v \in V;$$

here, λ is an eigenvalue of (Q) [or is in the spectrum of (Q)] if there exists a nonzero $u \in V$ such that the last equation holds.

REMARK. In the example of §3, we consider $H = L^2(\Omega)$, $a(u, u) = \|u\|_{H^1(\Omega)}^2$, $b(u, u) = \int_\Omega r|u|^2$ and $V = H_0^1(\Omega)$ [resp., $V = H^1(\Omega)$] for the Dirichlet [resp., Neumann] boundary value problem on an open set Ω with finite volume (resp., on a bounded open set Ω). In this manner, Theorems 3.1 and 3.2 will become corollaries of Theorems 4.1 and 4.2, respectively.

By (4.2), a induces in V an inner product equivalent to the original one; by the Riesz representation theorem, we define a bounded selfadjoint operator T from $(V, a(\cdot, \cdot))$ to itself by

$$(4.4) \quad b(u, v) = a(Tu, v), \quad \forall (u, v) \in V \times V.$$

It follows that λ is an eigenvalue of (Q) if and only if $1/\lambda$ is an eigenvalue of $T: Tu = (1/\lambda)u$, for some nonzero $u \in V$.

(4.5) We now assume that T is compact. This is the case in all the problems studied here; it holds in particular if V is compactly embedded in H and b is continuous on H .

It follows from the classical spectral theory of compact selfadjoint operators in Hilbert spaces that the spectrum of (Q) consists of a double sequence

$$\cdots \leq \lambda_{n+1}^- \leq \lambda_n^- \leq \cdots \leq \lambda_1^- < (0) \leq \lambda_1^+ \leq \cdots \leq \lambda_n^+ \leq \lambda_{n+1}^+ \leq \cdots$$

(each eigenvalue has finite multiplicity and is repeated accordingly).

When it exists, λ_n^\pm —which we write $\lambda_n^\pm(a, V)$ or $\lambda_n^\pm(b, V)$ if necessary—is given by the “max-min principle”

$$(4.6) \quad \frac{1}{\lambda_n^+} = \max_{F_n \in \mathcal{F}_n} \min_{u \in F_n} \{b(u, u) : a(u, u) = 1\},$$

where \mathcal{F}_n is the set of n -dimensional subspaces of V . A similar formula holds for λ_n^- since $\lambda_n^-(V, -b) = -\lambda_n^+(V, b)$.

Just as before, we deduce from (4.6) the following abstract monotonicity principles:

LEMMA 4.1. *If b_1, b_2 are two hermitian forms satisfying (4.3) and such that $b_1(u, u) \leq b_2(u, u)$ for all $u \in V$, then $\lambda_n^+(b_1, V) \geq \lambda_n^+(b_2, V)$.*

LEMMA 4.2. *Let a_1, a_2 be coercive forms satisfying (4.2) and such that $a_1(u, u) \geq a_2(u, u)$ for all $u \in V$; then $\lambda_n^+(a_1, V) \geq \lambda_n^+(a_2, V)$.*

LEMMA 4.3. *If (V_1, H, a) and (V_2, H, a) are two “variational triples” such that $V_1 \subset V_2$, then $\lambda_n^+(b, V_1) \geq \lambda_n^+(b, V_2)$.*

B. Operators of order $2m$. In the following, we suppose that Ω is a nonempty open subset of \mathbf{R}^k and that r and Ω satisfy (2.2); further the operator $\mathcal{A} = \sum_{|\alpha| \leq m, |\beta| \leq m} D^\alpha (a_{\alpha\beta} D^\beta)$ fulfills (2.1).

We first consider the Dirichlet boundary value problem: $\mathcal{A}u = \lambda ru$, $u \in H_0^m(\Omega)$.

(4.7) We assume that Ω has finite volume and that $|\Omega_+ \setminus \Omega_+^\circ| = 0$ and $|\Omega_- \setminus \Omega_-^\circ| = 0$.

We can now apply the above abstract theory with $H = L^2(\Omega)$, $V = H_0^m(\Omega)$ and

$$a(u, v) = \int_{\Omega} \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} a_{\alpha\beta} D^\alpha u \overline{D^\beta v}.$$

It follows from standard results on Sobolev spaces and from the uniform ellipticity of \mathcal{A} (i.e., $\mathcal{A}'(x, \xi) \geq c|\xi|^{2m}$, for all $\xi \in \mathbf{R}^k$ and some $c > 0$) that a satisfies (4.2).

By (2.2) and the Sobolev imbedding theorem [Ad, Theorem 5.4, p. 97], we see that $b(u, v) = \int_{\Omega} ru \overline{v}$ satisfies (4.3).

When $k > 2m$, we also have, by using the Sobolev and the Hölder inequalities that, for $u, v \in V$

$$\left| \int_{\Omega} ru \overline{v} \right| \leq \|r\|_{L^p(\Omega)} \|u\|_{L^s(\Omega)} \|v\|_{L^{2^*}(\Omega)},$$

where $2^* = 2k/(k - 2m)$ and $1/p + 1/s + 1/2^* = 1$.

Therefore, (4.5) follows from the Rellich-Kondrachov theorem [Ad, Theorem 6.2, p. 144]. Similar results are obtained when $k \leq 2m$ by use of [Ad, Theorem 5.4, Cases B and C, p. 97 and Theorem 6.2, equations (4) and (6), p. 144]. Much like in [dF, Proof of Proposition 1.11, p. 43], one then shows that the assumption $|\Omega_{\pm}| > 0$ implies that $|\lambda_n^{\pm}| \rightarrow +\infty$ as $n \rightarrow \infty$.

We now state

THEOREM 4.1. *Under hypotheses (2.1), (2.2) and (4.7), estimates (2.3) and (2.4) hold and*

$$(4.8) \quad |\lambda_n^{\pm}| \sim n^{2m/k} \left(\int_{\Omega_{\pm}} r^{k/2m}(x) \mu'_{\mathcal{A}}(x) dx \right)^{-2m/k}, \quad \text{as } n \rightarrow \infty.$$

PROOF. Estimate (2.3) holds when r is continuous and bounded away from zero (see [Me, Theorem 5.12, p. 188]); hence, for r nonnegative, we obtain the same

result by replacing r by $r + \delta$ and letting $\delta \rightarrow 0$. In light of Lemmas 4.1 and 4.3, we then obtain (4.8) and (2.3) exactly as in the proof of Theorem 3.1. Note that

$$\int_{\Omega_{\pm}} r^{k/2m} \mu'_{\mathcal{A}} = \int_{\Omega_{\pm}^{\circ}} r^{k/2m} \mu'_{\mathcal{A}} = \int_{\Omega} (r_{\pm})^{k/2m} \mu'_{\mathcal{A}}. \quad \square$$

Likewise, we have

THEOREM 4.2. *For the Neumann problem, if r, Ω and \mathcal{A} satisfy (2.1), (2.2) and (3.7), then (2.3), (2.4) and (4.8) hold.*

PROOF. We now consider $V = H^m(\Omega)$; we use, in particular, Lemma 4.3 with $V_1 = H_0^m(\Omega)$ and $V_2 = H^m(\Omega)$; and the proof goes through as in Theorems 3.2 and 4.1. \square

5. The Schrödinger operator. In this section, we give sufficient conditions under which a positive operator of “Schrödinger type” and with an indefinite weight has discrete “spectrum” in an unbounded open set $\Omega \subset \mathbf{R}^k$ and we then determine the asymptotic distribution of its eigenvalues. We combine, in particular, the results and methods of §4 and previous works [Ro 2; Ro 3; Fl; etc.] on Schrödinger operators with positive weights.

DEFINITION 5.1. *Let f be a continuous function defined on an open subset ω of \mathbf{R}^k . We say that f satisfies (ε) if there exists a positive number ε_0 such that f can be continuously extended to $\tilde{\omega} = \{x \in \mathbf{R}^k : \text{dist}(x, \omega) < \varepsilon_0\}$ and, for all $\varepsilon \in (0, \varepsilon_0)$, there exists $\eta > 0$ such that, for all $(x, y) \in \tilde{\omega} \times \tilde{\omega}$ with $|x - y| \leq \eta$, we have $|f(x) - f(y)| \leq \varepsilon |f(x)|$.*

REMARKS. 1. For instance, $f(x) = (1 + |x|^2)^{\sigma}$, with $\sigma \geq 1$, satisfies (ε) .

2. If a positive function f satisfies (ε) , so does its inverse $1/f$.

In the following, we set $\Omega'_{\rho} = \{x \in \Omega : |x| > \rho\}$ for $\rho > 0$.

(5.1) Let q be a nonnegative function in $L_{\text{loc}}^{k/2m}(\Omega)$, tending to $+\infty$ at infinity and satisfying (ε) on $\Omega'_{R_1} \cap \Omega_+$, for some $R_1 > 0$.

(5.2) Let r be a function defined on Ω such that $q^{-1}r$ tends to 0 at infinity. Moreover, we suppose that

(a) Assumption (2.2) holds and $|\Omega_+ \setminus \Omega_+^{\circ}| = 0$.

(b) The restriction of r to $\Omega'_{R_1} \cap \Omega_+$ satisfies (ε) .

(c) $\int_{\Omega} (r_+)^{k/2m} = +\infty$.

(Note that r is allowed to change sign anywhere in Ω .)

REMARK. (5.2) holds, for example, if $\Omega = \mathbf{R}^k$ with $r(x) = -1$ if $|x| \leq 1$ and $r(x) = +1$ if $|x| > 1$.

(5.3) Let \mathcal{L} be a formally selfadjoint, uniformly elliptic operator of order $2m$ defined on Ω and verifying (2.1):

$$\mathcal{L} = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D^{\alpha} (a_{\alpha\beta} D^{\beta});$$

we suppose that for all $|\alpha| = |\beta| = m$, $a_{\alpha\beta}$ satisfies (ε) on Ω'_{R_1} ; in addition, we assume that there exists $d > 0$ such that for all open subsets ω of Ω ,

$$\int_{\omega} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha} u \overline{D^{\beta} u} \geq d \|u\|_{H^m(\omega)}^2, \quad \forall u \in H^m(\omega).$$

DEFINITION 5.2. If ω is an open subset of \mathbf{R}^k , $V_q^0(\omega)$ denotes the completion of $C_0^\infty(\omega)$ with respect to the Hilbert norm

$$\|u\|_{V_q(\omega)} = \left(\|u\|_{H^m(\omega)}^2 + \int_{\omega} q|u|^2 \right)^{1/2}.$$

The space $V_q^1(\omega)$ is the set of restrictions to ω of elements of $V_q^0(\Omega)$.

We study the variational eigenvalue problem

$$(S) \quad \mathcal{A}u = (\mathcal{L} + q)u = \lambda ru, \quad \text{in } \Omega, \quad \text{with Dirichlet boundary conditions.}$$

In other words, λ is an eigenvalue of (S) if there exists a nonzero u in $V_q^0(\Omega)$ such that the above equation holds in the distributional sense.

We now apply the abstract theory developed in §4.A; here, $H = L^2(\Omega)$ is equipped with its usual inner product (\cdot, \cdot) , $V = V_q^0(\Omega)$, $b(u, u) = (ru, u)$ and $a(u, u) = (\mathcal{L}u, u) + (qu, u)$, for $u \in V$.

It follows from (5.1) and (5.3) that a is coercive on $V_q^0(\Omega)$; further, the fact that q tends to infinity at infinity implies that the imbedding of $V_q^0(\Omega)$ into $L^2(\Omega)$ is compact. Moreover, (4.5) holds because, for $\rho > 0$,

$$(5.4) \quad \left| \int_{\Omega'_\rho} r|u|^2 \right| \leq \left(\sup_{x \in \Omega'_\rho} \left| \frac{r}{q} \right| \right) \left(\int_{\Omega'_\rho} q|u|^2 \right) \leq \delta(\rho) \|u\|_{V_q(\Omega'_\rho)}^2,$$

where $\delta(\rho)$ tends to zero as $\rho \rightarrow +\infty$; this follows since, by (5.2), r/q tends to zero at infinity.

Consequently, the spectrum of (S) is discrete; moreover, the eigenvalues of (S) are characterized by the “max-min” principle” (4.6).

Let $\lambda_n^+(r, \omega)$ [resp., $\Lambda_n^+(r, \omega)$] denote the nonnegative eigenvalues of the variational Dirichlet [resp., Neumann] boundary value problem

$$\mathcal{A}u = \lambda ru, \quad \text{in } \omega \subset \Omega; \quad u \in V_q^0(\omega) \quad [\text{resp., } u \in V_q^1(\omega)].$$

Set $N_0^+(\lambda; r, \omega, \mathcal{A}) = \sum_{\lambda_n^+(r, \omega) < \lambda} 1$ and $N_1^+(\lambda; r, \omega, \mathcal{A}) = \sum_{\Lambda_n^+(r, \omega) < \lambda} 1$; one indicates in this way the dependence on the weight function r , the open set ω and the operator \mathcal{A} .

Note that

$$N_0^+(\lambda; r, \Omega, \mathcal{A}) = N^+(\lambda),$$

where $N^+(\lambda)$ stands for the number of nonnegative eigenvalues of (S) less than $\lambda > 0$.

We shall use the subsequent lemma, which is derived from Lemma 4.3.

LEMMA 5.1. If ω_1 and ω_2 are two disjoint open sets in $\omega \subset \Omega$ such that $\bar{\omega}_1 \cup \bar{\omega}_2 = \bar{\omega}$, then

$$\begin{aligned} N_0^+(\lambda; r, \omega_1, \mathcal{A}) + N_0^+(\lambda; r, \omega_2, \mathcal{A}) &\leq N_0^+(\lambda; r, \omega, \mathcal{A}) \leq N_1^+(\lambda; r, \omega, \mathcal{A}) \\ &\leq N_1^+(\lambda; r, \omega_1, \mathcal{A}) + N_1^+(\lambda; r, \omega_2, \mathcal{A}). \end{aligned}$$

This is due in particular to the inclusions

$$V_q^0(\omega_1) \oplus V_q^0(\omega_2) \subset V_q^0(\omega) \quad \text{and} \quad V_q^1(\omega) \subset V_q^1(\omega_1) \oplus V_q^1(\omega_2).$$

Let us consider a covering of \mathbf{R}^k by a countable family of disjoint open cubes $\{Q_\zeta\}_{\zeta \in \mathbf{Z}^k}$ with center x_ζ and sides of length η :

$$\mathbf{R}^k = \bigcup_{\zeta \in \mathbf{Z}^k} \overline{Q}_\zeta.$$

Let $R > R_1$. Set $B = \{x \in \Omega: |x| < R\}$ and $G = \Omega'_R = \{x \in \Omega: |x| > R\}$; write

$$G_+ = \{x \in G: r(x) > 0\} \quad \text{and} \quad G_- = \{x \in G: r(x) < 0\}.$$

Fix $\lambda > 0$; let $G_\lambda = \{x \in G: \lambda r(x) - q(x) > 0\}$.

We note that G_+ and G_λ are open; moreover, we have

$$G_\lambda \subset G_+ \quad \text{and} \quad \text{dist}(G_\lambda, G \setminus G_+) > 0;$$

this follows since $r \geq \gamma/\lambda$ on G_λ and r, q are continuous on G_+ ; here and thereafter, we choose R_1 (and $R > R_1$) so large that $q \geq \gamma$ on Ω'_{R_1} for some positive constant γ ; this is possible in view of (5.1). Since, by (5.2), G_λ is bounded, the sets I and J that we now introduce are finite.

Put $I = \{\zeta \in \mathbf{Z}^k: \overline{Q}_\zeta \subset G_\lambda\}$ and $J = \{\zeta \in \mathbf{Z}^k: \overline{Q}_\zeta \cap \overline{G}_\lambda \neq \emptyset\}$. We choose η so small that $(\bigcup_{\zeta \in J} \overline{Q}_\zeta) \subset G_+$.

We further make some technical assumptions on r, q and G_λ (recall that r is positive on G_λ):

$$(5.5) \quad \lim_{\eta \rightarrow 0} \left(\sum_{\zeta \in J \setminus I} (r_\zeta)^{k/2m} \right) \left(\sum_{\zeta \in I} (r_\zeta)^{k/2m} \right)^{-1} = 0,$$

where $r_\zeta = r(x_\zeta)$ for $\zeta \in \mathbf{Z}^k$.

(5.6) Finally, we suppose that there exist $\lambda' > 0$ and $c > 0$ such that G_λ is Jordan contented (see [LoS, Chapter 8]) and $[G_\lambda] \leq c[G_{\lambda/2}]$, $\forall \lambda \geq \lambda'$, where $[G_\lambda] = \int_{G_\lambda} r^{k/2m}$. Intuitively, (5.5) means that ∂G_λ is of zero measure with respect to $r(x) dx$; moreover, hypothesis (5.6) is of Tauberian type.

We can now state

THEOREM 5.1. *Under hypotheses (5.1)–(5.3) and (5.5)–(5.6), the spectrum of (S) is discrete and estimate (2.7) holds:*

$$N^+(\lambda) \sim \int_{\Omega_+} [(\lambda r - q)_+]^{k/2m} \mu'_\mathcal{L}, \quad \text{as } \lambda \rightarrow +\infty,$$

where $\Omega_+ = \{x \in \Omega: r(x) > 0\}$.

The idea of the proof consists in breaking \mathbf{R}^k into disjoint pieces and linking the estimates so obtained with the help of Lemma 5.1. Inside the bounded open set B , we apply the results of §4.B, where r is allowed to be “singular”; outside B , we reduce the problem to G_λ on which r is “smooth” and positive and a method similar to that of [F1] can be used.

To prove Theorem 5.1, we shall need the following results:

PROPOSITION 5.1. *There exist two positive constants c' and c'' such that, for all λ sufficiently large,*

$$c' \lambda^{k/2m} [G_\lambda] \leq \varphi(\lambda) \leq c'' \lambda^{k/2m} [G_\lambda],$$

where $\varphi(\lambda) = \int_{\Omega_+} [(\lambda r - q)_+]^{k/2m} \mu'_\mathcal{L}$.

PROOF OF PROPOSITION 5.1. It follows from (5.1)–(5.3) that

$$\varphi(\lambda) = \int_{G_\lambda} (\lambda r - q)^{k/2m} \mu'_{\mathcal{L}} \leq c'' \int_{G_\lambda} (\lambda r)^{k/2m}$$

and from (5.6) that

$$\int_{G_\lambda} (\lambda r - q)^{k/2m} \mu'_{\mathcal{L}} \geq \int_{G_{\lambda/2}} (\lambda r - q)^{k/2m} \mu'_{\mathcal{L}} \geq c' \int_{G_\lambda} (\lambda r)^{k/2m};$$

note that $r = r_+$ and $\lambda r - q = (\lambda r - q)_+$ on $G_\lambda \supset G_{\lambda/2}$. \square

PROPOSITION 5.2. *If Theorem 5.1 holds for $N_0^+(\lambda; r, \Omega, \mathcal{A}')$, then it holds for $N_0^+(\lambda; r, \Omega, \mathcal{A})$, where $\mathcal{A}' = \mathcal{L}' + q$ and \mathcal{L}' is the principal part of \mathcal{L} .*

PROOF OF PROPOSITION 5.2. This is a simple consequence of interpolation theorems and of Lemma 4.2 if we notice—as in [Ro 2, Lemma 1.1, p. 353]—that

$$\varphi((1 + \varepsilon)\lambda) \leq (1 + c_1 \sqrt{\varepsilon})^{k/2m} \varphi(\lambda),$$

for some positive constant c_1 .

The result now follows by letting $\varepsilon \rightarrow 0$; here and thereafter, ε denotes the variable used in Definition 5.1. \square

PROPOSITION 5.3.

$$\begin{aligned} \lim_{\lambda \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \varphi^{-1}(\lambda) \sum_{\varsigma \in J} \varphi^+(\lambda, \varsigma) &= 1 \\ &= \lim_{\lambda \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \varphi^{-1}(\lambda) \sum_{\varsigma \in I} \varphi^-(\lambda, \varsigma), \end{aligned}$$

where

$$\begin{aligned} \varphi^\pm(\lambda, \varsigma) &= [(1 \pm \varepsilon)\lambda r_\varsigma - q_\varsigma]^{k/2m} \mu_\varsigma |Q_\varsigma|, \\ q_\varsigma &= q(x_\varsigma) \quad \text{and} \\ \mu_\varsigma &= (2\pi)^{-k} \int_{\{\xi \in \mathbf{R}^k; \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_\varsigma) \xi^{\alpha+\beta} < 1\}} d\xi. \end{aligned}$$

This follows easily from hypothesis (5.5); observe that $\varepsilon \rightarrow 0$ implies that $\eta \rightarrow 0$.

PROOF OF THEOREM 5.1. In light of Proposition 5.2, it suffices to establish Theorem 5.1 for \mathcal{A}' .

Now, we make use of the covering $(Q_\varsigma)_{\varsigma \in \mathbf{z}^k}$ of \mathbf{R}^k and of Lemma 5.1:

$$\begin{aligned} (5.7) \quad \sum_{\varsigma \in I} N_0^+(\lambda; r, Q_\varsigma, \mathcal{A}') &\leq N_0^+(\lambda; r, \Omega, \mathcal{A}') \\ &\leq N_1^+(\lambda; r, B, \mathcal{A}') + \sum_{\varsigma \in J} N_1^+(\lambda; r, Q_\varsigma, \mathcal{A}') \\ &\quad + N_1^+(\lambda; r, D, \mathcal{A}') + N_1^+(\lambda; r, G \setminus \overline{G}_+, \mathcal{A}'), \end{aligned}$$

where $D = G_+ \setminus (\bigcup_{\varsigma \in J} \overline{Q}_\varsigma)$.

Next, we remark that $N_1^+(\lambda; r, G \setminus \overline{G}_+, \mathcal{A}') = 0$ because on $G \setminus \overline{G}_+$, r is nonpositive and, by (4.6), there are no positive eigenvalues. Moreover, $N_1^+(\lambda; r, D, \mathcal{A}') = 0$; the

latter equality holds since $D \subset U_\lambda$, where $U_\lambda = \{x \in G_+ : \lambda r(x) - q(x) < 0\}$; hence $(\int_{U_\lambda} q|u|^2 / \int_{U_\lambda} r|u|^2) \geq \lambda$ and, by (4.6), $N_1^+(\lambda; r, U_\lambda, \mathcal{A}') = 0$.

By Theorems 4.1 and 4.2, we have

$$(5.8) \quad N_i^+(\lambda; r, B, \mathcal{A}') \sim \int_{B_+} [(\lambda r - q)_+]^{k/2m} \mu'_\mathcal{L}, \quad \text{as } \lambda \rightarrow +\infty,$$

where $B_+ = \{x \in B : r(x) > 0\}$ and $i = 0$ or 1 according to the boundary conditions; in fact, by (2.2), we may choose R big enough so that $|B_+| > 0$; moreover, we observe that, because B is bounded,

$$\int_{B_+} (\lambda r)^{k/2m} \mu'_\mathcal{L} \sim \int_{B_+} [(\lambda r - q)_+]^{k/2m} \mu'_\mathcal{L}, \quad \text{as } \lambda \rightarrow +\infty.$$

It follows, in particular, that $\varphi^{-1}(\lambda)N_i(\lambda; r, B, \mathcal{A}') \rightarrow 0$ as $\lambda \rightarrow +\infty$.

Thanks to (5.7), we only have to work on $\bigcup_{\zeta \in J} Q_\zeta$, where r is positively bounded from below by $\lambda^{-1}\gamma$ and satisfies (ε) . On each cube Q_ζ , where $\zeta \in J$, we can compare \mathcal{A}' with a homogeneous operator with constant coefficients; precisely, for $i = 0$ or 1 , it follows from condition (ε) and from Lemmas 4.1 and 4.2 that, for η small enough,

$$(5.9) \quad \begin{aligned} N_i^+([(1-\varepsilon)\lambda - q_\zeta/r_\zeta]; r_\zeta, Q_\zeta, \mathcal{L}_\zeta) &\leq N_i^+(\lambda; r, Q_\zeta, \mathcal{A}') \\ &\leq N_i^+([(1+\varepsilon)\lambda - q_\zeta/r_\zeta]; r_\zeta, Q_\zeta, \mathcal{L}_\zeta), \end{aligned}$$

where $\mathcal{L}_\zeta = \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(x_\zeta) D^{\alpha+\beta}$ and, as above, $r_\zeta = r(x_\zeta)$; furthermore, we know by a special case of [Me, Theorem 5.1, p. 175 and Theorem 5.12, p. 188] that there exists C independent of ζ such that

$$(5.10) \quad \begin{aligned} |N_i^+([(1 \pm \varepsilon)\lambda - q_\zeta/r_\zeta]; r_\zeta, Q_\zeta, \mathcal{L}_\zeta) - \mu_\zeta |Q_\zeta| [(1 \pm \varepsilon)\lambda r_\zeta - q_\zeta]^{k/2m}| \\ \leq C(\lambda r_\zeta)^{(k-1)/2m}. \end{aligned}$$

In light of Lemma 5.1, Propositions 5.1–5.3 and equations (5.7)–(5.10), we conclude the proof of Theorem 5.1 by choosing $\varepsilon = \lambda^{-1/2(2m-1)}$ and by letting $\lambda \rightarrow +\infty$. \square

The next statement is really a corollary of the proof of Theorem 5.1 although it is an extension of Weyl's formula (Theorem 4.1).

THEOREM 5.2. *If in assumption (5.2)(c) we suppose instead that $\int_\Omega |r|^{k/2m}$ is finite, we must replace the conclusion of Theorem 5.1 by that of Theorem 4.1:*

$$N^+(\lambda) \sim \int_{\Omega_+} (\lambda r)^{k/2m} \mu'_\mathcal{L}, \quad \text{as } \lambda \rightarrow +\infty.$$

REMARKS. 1. Of course, analogous estimates hold for negative eigenvalues provided that hypotheses (5.1), (5.2) and (5.5), (5.6) are changed accordingly.

It is noteworthy that the value of $\int_\Omega (r_-)^{k/2m}$ does not interfere with the asymptotics of $N^+(\lambda; \mathcal{A}, r, \Omega)$ and vice versa. We give an example below where $N^+(\lambda)$ and $N^-(\lambda)$ do not have the same estimate (one is given by Theorem 5.1 and the other by Theorem 5.2). We note that, in the case when $\int_\Omega (r_\pm)^{k/2m}$ are both finite, Theorem 5.2 (and its analogue for N^-) has been proved under slightly different hypotheses by Rozenbljum [Ro 1, Theorem 2, p. 246].

2. A result similar to Theorems 5.1 and 5.2 could also be obtained by the same methods for mixed Dirichlet-Neumann problems (under suitable compactness hypotheses).

3. Recently, Gurarie [Gu] has studied the asymptotic distribution of the eigenvalues of operators of Schrödinger type in \mathbf{R}^k having a smooth *positive* weight function; in his work, based in part on the theory of pseudodifferential operators, the analogue of condition (ε) is the so-called “finite propagation speed” condition.

The following result is of interest for certain applications:

PROPOSITION 5.4. *The conclusions of Theorems 5.1 and 5.2 remain valid if we drop the hypothesis that q tends to $+\infty$ at infinity and replace it by the assumption that r is positive outside B .*

In this case, as in the abstract part of §4, the imbedding of V into H is continuous (but not compact); however T is compact by (5.4) since we still assume that $q^{-1}r$ tends to 0 at infinity.

EXAMPLE 5.1. We now illustrate our results by considering the following eigenvalue problem:

$$(ES) \quad \mathcal{A}u = (-\Delta + q)u = \lambda ru, \quad \text{in } \mathbf{R}^k,$$

where

$$r(x) = \begin{cases} -1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| = 1, \\ +1 & \text{if } |x| > 1, \end{cases}$$

and where q satisfies (5.1) [e.g., $q(x) = (1 + |x|^2)^\sigma$ with $\sigma \geq 1$].

According to Theorem 5.1, Theorem 5.2 and the preceding Remark 1, we then have

COROLLARY 5.1.

$$N^-(\lambda) \sim (2\pi)^{-k} (B_k)^2 |\lambda|^{k/2}, \quad \text{as } \lambda \rightarrow -\infty,$$

and

$$N^+(\lambda) \sim (2\pi)^{-k} B_k \int_{|x|>1} [(\lambda - q(x))_+]^{k/2} dx, \quad \text{as } \lambda \rightarrow +\infty,$$

where B_k denotes the volume of the unit ball in \mathbf{R}^k .

NOTE ADDED IN PROOF. Because of an erroneous numerical choice of the sharp Sobolev constant [LY, equation (19), p. 316], the claim in [LiY, Theorem 2 and Corollary 2] that the constant found there is better than in [Lb] is not justified. The best constant for the Cwikel-Lieb-Rozenbljum bound seems still due to Lieb in [Lb], as is clearly discussed on page 475 of E. H. Lieb, *On characteristic exponents in turbulence*, Comm. Math. Phys. **92** (1984), 473–480. In §3.C of the present paper, one should therefore use [Lb, Theorem 2, p. 243] rather than [LiY, Theorem 2, p. 314]; more specifically, if L_k and \tilde{C}_k are defined as in [Lb, p. 243] and [LiY, p. 310], respectively, one must replace Θ_k by $\tilde{C}_k = (L_k)^{-2/k}$ in equation (3.10) of Theorem 3.3 and substitute $L_k^{-1}N^\pm(\lambda)$ for the left-hand side of the inequality in Corollary 3.2.

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